SOLUTION OF A CONTACT PROBLEM FOR AN ELASTIC RECTANGLE

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The plane contact problem of the theory of elasticity for a semi-infinite plane, a strip and a quarter-plane has been studied in [1 to 9] and others.

In the present paper the solution of a contact problem for an elastic rectangle is reduced to a quasi-completely regular infinite system of linear algebraic equations with finite absolute terms.

The theory of infinite systems of linear algebraic equations has been successfully applied to the solution of elasticity problems by Koialovich [10], Kantorovich [11], Arutiunian [12] and others.

1. Consider the plane problem of the theory of elasticity for a rectangle (Fig. 1) with symmetrical boundary conditions when the external load is specified on the part of the longer sides of the rectangle ($-2a-l \leqslant x \leqslant -2a,\ 0 \leqslant x \leqslant l$) and when the normal displacements and shear stresses are given on the remaining part of the same sides (i.e. under the die). The shear stresses and normal displacements are also specified on the sides $x=-2a-l$ and $x=l$. In the particular case when these stresses and displacements on the lateral edges and the shear stresses on the longitudinal edges of the rectangle are zero, we have the periodic contact problem for a strip.

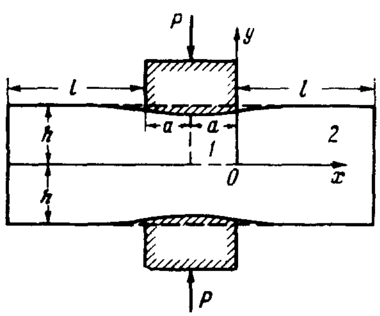

FIG. 1

We seek a solution to the problem by using an Airy stress function, which in the absence of mass forces satisfies the biharmonic equation within the rectangle $-(2a+l) \leqslant x \leqslant l,\ -h \leqslant y \leqslant h$.

The stresses and displacements can be expressed in terms of the stress function Φ as follows:

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2\Phi}{\partial x^2} \tag{1.1}$$

$$u = \frac{1}{E}\left\{\int \frac{\partial^2\Phi}{\partial y^2}\,dx - \nu\frac{\partial\Phi}{\partial x}\right\} - a^* y + b^*$$
$$v = \frac{1}{E}\left\{\int \frac{\partial^2\Phi}{\partial x^2}\,dy - \nu\frac{\partial\Phi}{\partial y}\right\} + a^* x + c^* \tag{1.2}$$

Here E is the modulus of elasticity, ν is Poisson's ratio and a^*, b^* and c^* are the constants which determine the rigid-body displacement of the elastic body. By virtue of the symmetry of the boundary conditions about the axes of symmetry of the rectange we need to determine the function $\Phi(x, y)$ only for the first quadrant of the rectangle.

We denote the stress function Φ in the regions *1* and *2* by Φ_1 and Φ_2, respectively

$$\Phi(x, y) = \begin{cases}\Phi_1(x, y) & \text{in region } 1\\ \Phi_2(x, y) & \text{in region } 2\end{cases} \tag{1.3}$$

We seek the biharmonic functions Φ_1 and Φ_2 in the form [13 and 14]

$$\Phi_1(x, y) = \frac{1}{2}(A_1x^2 + B_1y^2) + \sum_{k=1}^{\infty}[A_k^{(1)}\cosh\alpha_k y + B_k^{(1)}\alpha_k y\sinh\alpha_k y]\cos\alpha_k x +$$
$$+\sum_{k=1}^{\infty}[C_k^{(1)}\cosh\beta_k x + D_k^{(1)}\sinh\beta_k x + \beta_k x(E_k^{(1)}\cosh\beta_k x + F_k^{(1)}\sinh\beta_k x)]\cos\beta_k y$$
$$(-a<x<0,\ 0<y<h) \tag{1.4}$$
$$\Phi_2(x, y) = \frac{1}{2}(A_2x^2 + B_2y^2) + \sum_{k=1}^{\infty}[A_k^{(2)}\cosh\gamma_k y + B_k^{(2)}\gamma_k y\sinh\gamma_k y]\cos\gamma_k x +$$
$$+\sum_{k=1}^{\infty}[C_k^{(2)}\cosh\beta_k x + D_k^{(2)}\sinh\beta_k x + \beta_k x(E_k^{(2)}\cosh\beta_k x + F_k^{(2)}\sinh\beta_k x)]\cos\beta_k y$$

where

$$(0<x<l,\ 0<y<h) \tag{1.5}$$
$$\alpha_k = \frac{k\pi}{a}, \qquad \beta_k = \frac{k\pi}{h}, \qquad \gamma_k = \frac{k\pi}{l} \tag{1.6}$$

Because of the symmetry of the boundary conditions we have

$$\tau_{xy}(x, 0) = 0, \quad v(x, 0) = 0 \qquad (-a<x<l) \tag{1.7}$$
$$\tau_{xy}(-a, y) = 0, \quad u(-a, y) = 0 \qquad (0<y<h) \tag{1.8}$$

The following conditions hold on the contour of the rectangle:

$$\tau_{xy}(l, y) = \sum_{k=1}^{\infty} c_k\sin\beta_k y, \qquad u(l, y) = \frac{d_0}{2} + \sum_{k=1}^{\infty} d_k\cos\beta_k y \qquad (0<y<h) \tag{1.9}$$
$$v(x, h) = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k\cos\alpha_k x, \qquad \tau_{xy}(x, h) = \sum_{k=1}^{\infty} a_k\sin\alpha_k x \qquad (-a<x<0) \tag{1.10}$$
$$\sigma_y(x, h) = \frac{f_0}{2} + \sum_{k=1}^{\infty} f_k\cos\gamma_k x, \quad \tau_{xy}(x, h) = \sum_{k=1}^{\infty} e_k\sin\gamma_k x \qquad (0<x<l) \tag{1.11}$$

In addition, on the side common to regions *1* and *2* the conditions of continuity of the stresses τ_{xy} and σ_x and of the displacements u and v must be satisfied. They are

$$\tau_{xy}^1(0, y) = \tau_{xy}^2(0, y), \qquad u^1(0, y) = u^2(0, y) \qquad (0 < y < h) \tag{1.12}$$

$$\sigma_x^1(0, y) = \sigma_x^2(0, y), \qquad v^1(0, y) = v^2(0, y) \tag{1.13}$$

The superscript indicates from the side of which region the quantities are calculated.

2. Substituting (1.4) and (1.5) into (1.7), we note that the first condition of (1.7) is satisfied identically. From the second condition we have that

$$a^* = c^* = 0 \tag{2.1}$$

Satisfying conditions (1.8) to (1.11) according to (1.4) to (1.6) and (2.1), we obtain

$$A_1 = \frac{Eb_0}{2h} + \nu B_1, \qquad b_1^* = \frac{a}{E}(B_1 - \nu A_1), \qquad b_2^* = \frac{d_0}{2} - \frac{l}{E}(B_2 - \nu A_2) \tag{2.2}$$

$$(A_k^{(1)} + B_k^{(1)}) \sinh \alpha_k h + B_k^{(1)} \alpha_k h \cosh \alpha_k h = \frac{a_k}{\alpha_k^2}$$

$$(A_k^{(1)} - B_k^{(1)}) \sinh \alpha_k h + B_k^{(1)} \alpha_k h \cosh \alpha_k \cosh = -\frac{\nu a_k}{\alpha_k^2} - \frac{Eb_k}{\alpha_k} \tag{2.3}$$

$$-(C_k^{(1)} + F_k^{(1)}) \sinh \beta_k a + (D_k^{(1)} + E_k^{(1)}) \cosh \beta_k a + \beta_k a (E_k^{(1)} \sinh \beta_k a - F_k^{(1)} \cosh \beta_k a) = 0$$

$$-(C_k^{(1)} - F_k^{(1)}) \sinh \beta_k a + (D_k^{(1)} - E_k^{(1)}) \cosh \beta_k a + \beta_k a (E_k^{(1)} \sinh \beta_k a - F_k^{(1)} \cosh \beta_k a) = 0 \tag{2.4}$$

$$(C_k^{(2)} + F_k^{(2)}) \sinh \beta_k l + (D_k^{(2)} + E_k^{(2)}) \cosh \beta_k l + \beta_k l (E_k^{(2)} \sinh \beta_k l + F_k^{(2)} \cosh \beta_k l) = \frac{c_k}{\beta_k^2} \tag{2.5}$$

$$(C_k^{(2)} - F_k^{(2)}) \sinh \beta_k l + (D_k^{(2)} - E_k^{(2)}) \cosh \beta_k l + \beta_k l (E_k^{(2)} \sinh \beta_k l + F_k^{(2)} \cosh \beta_k l) = -\frac{\nu c_k}{\beta_k^2} - \frac{Ed_k}{\beta_k}$$

$$(A_k^{(2)} + B_k^{(2)}) \sinh \gamma_k h + B_k^{(2)} \gamma_k h \cosh \gamma_k h = \frac{e_k}{\gamma_k^2} \tag{2.6}$$

From (1.4), (1.5) and conditions (1.12) we obtain

$$D_k^{(1)} + E_k^{(1)} = D_k^{(2)} + E_k^{(2)}, \qquad D_k^{(1)} - E_k^{(1)} = D_k^{(2)} - E_k^{(2)}, \qquad b_1^* = b_2^* \tag{2.7}$$

Here b_1^* and b_2^* are the values of b^* in the regions *1* and *2*. We introduce the notations

$$B_k^{(2)} = \frac{h}{\gamma_k} \frac{X_k}{\sinh \gamma_k h}, \quad D_k^{(2)} = \frac{(-1)^k l}{\beta_k} Y_k, \quad E_k^{(2)} = \frac{(-1)^k l}{\beta_k} Z_k \tag{2.8}$$

Making use of (1.4), (1.5), (1.11) and (1.13) together with (2.1) to (2.7), we obtain after some rearrangement the following set of three infinite systems

$$X_p = \sum_{k=1}^{\infty} a_{pk} Y_k + \sum_{k=1}^{\infty} b_{pk} Z_k + m_p \qquad (p = 1, 2, 3, \ldots) \tag{2.9}$$

$$Y_p = \sum_{k=1}^{\infty} c_{pk} X_k + n_p, \qquad Z_p = \sum_{k=1}^{\infty} d_{pk} X_k + r_p \tag{2.10}$$

where

$$a_{pk} = -\frac{2\gamma_p}{h\zeta_p}\frac{1}{\beta_k^2+\gamma_p^2}, \qquad b_{pk} = \frac{2\gamma_p}{h\zeta_p}\frac{\beta_k^2-\gamma_p^2}{(\beta_k^2+\gamma_p^2)^2} \tag{2.11}$$

$$c_{pk} = -\frac{4\beta_p}{l\xi_p}\left(\frac{\gamma_k^2}{\gamma_k^2+\beta_p^2}+\frac{\eta_p}{2\xi_p}\right)\frac{1}{\gamma_k^2+\beta_p^2}, \qquad d_{pk} = -\frac{2\beta_p}{l\xi_p}\frac{1}{\gamma_k^2+\beta_p^2} \tag{2.12}$$

$$m_p = \frac{2}{\nu h^2\gamma_p\zeta_p}\sum_{k=1}^{\infty}\left(\frac{a_k}{\alpha_k}-\frac{e_k}{\gamma_k}\right)+\frac{2}{hl\gamma_p\zeta_p}\sum_{k=1}^{\infty}c_k\frac{(-1)^k}{\beta_k}\times$$
$$\times\left\{1-(-1)^p\frac{\beta_k^2[\beta_k^2+(2+\nu)\gamma_p^2]}{(\beta_k^2+\gamma_p^2)^2}\right\}-\frac{2E}{hl}\frac{(-1)^p\gamma_p}{\zeta_p}\sum_{k=1}^{\infty}d_k\frac{(-1)^k\beta_k^2}{(\beta_k^2+\gamma_p^2)^2}+ \tag{2.13}$$
$$+\frac{1}{h\gamma_p\zeta_p}\left[f_p+e_p\coth\gamma_p h-f_0-\frac{E}{l}\left(\frac{a}{h}b_0+\frac{d_0}{\nu}\right)\right]+\frac{2}{\gamma_p\zeta_p}\frac{l+(1-\nu^2)a}{\nu hl}B_1$$

$$n_p = \frac{2}{hl\beta_p\xi_p}\sum_{k=1}^{\infty}\frac{a_k\alpha_k}{\alpha_k^2+\beta_p^2}\left[\frac{\alpha_k^2+(2+\nu)\beta_p^2}{\alpha_k^2+\beta_p^2}-\frac{1+\nu}{2\nu}\frac{\eta_p}{\xi_p}\frac{\alpha_k^2+(1-\nu)\beta_p^2}{\alpha_k^2}\right]+$$
$$+\frac{2E}{hl}\frac{\beta_p}{\xi_p}\sum_{k=1}^{\infty}\frac{b_k}{\alpha_k^2+\beta_p^2}\left(\frac{\alpha_k^2}{\alpha_k^2+\beta_p^2}+\frac{\eta_p}{2\xi_p}\right)-\frac{2}{hl\beta_p\xi_p}\sum_{k=1}^{\infty}\frac{e_k}{\gamma_k}\left(\frac{\gamma_k^2}{\gamma_k^2+\beta_p^2}-\frac{1+\nu}{2\nu}\frac{\eta_p}{\xi_p}\right)+$$
$$+\frac{(-1)^p(1-\nu)}{2l\beta_p\xi_p\sinh\beta_p l}\left[\left(1-\frac{1+\nu}{1-\nu}\beta_p l\coth\beta_p l\right)c_p-\frac{E\beta_p}{1-\nu}(1+\beta_p l\coth\beta_p l)\,d_p\right]+$$
$$+\frac{(-1)^p(1+\nu)}{2l\beta_p\sinh\beta_p l}\frac{\eta_p}{\xi_p^2}\left(c_p+\frac{E\beta_p}{1+\nu}d_p\right)+\frac{E}{2l^2\beta_p}\frac{\eta_p}{\xi_p^2}\left(\frac{a+l}{h}b_0+\frac{d_0}{\nu}\right)-$$
$$-\frac{(1-\nu^2)(a+l)}{\nu l^2\beta_p}\frac{\eta_p}{\xi_p^2}B_1 \tag{2.14}$$

$$r_p = -\frac{1+\nu}{\nu hl\beta_p\xi_p}\sum_{k=1}^{\infty}\frac{a_k}{\alpha_k}\frac{\alpha_k^2+(1-\nu)\beta_p^2}{\alpha_k^2+\beta_p^2}+\frac{E}{hl}\frac{\beta_p}{\xi_p}\sum_{k=1}^{\infty}\frac{b_k}{\alpha_k^2+\beta_p^2}+$$
$$+\frac{1+\nu}{\nu hl\beta_p\xi_p}\sum_{k=1}^{\infty}\frac{e_k}{\gamma_k}+\frac{(-1)^p(1+\nu)}{2l\beta_p\xi_p\sinh\beta_p l}\left(c_p+\frac{E\beta_p}{1+\nu}d_p\right)+\frac{E}{2l^2\beta_p\xi_p}\left(\frac{a+l}{h}b_0+\frac{d_0}{\nu}\right)-$$
$$-\frac{(1-\nu^2)(a+l)}{\nu l^2\beta_p\xi_p}B_1 \tag{2.15}$$

$$\xi_p = \coth\beta_p a+\coth\beta_p l, \qquad \eta_p = \frac{\beta_p a}{\sinh^2\beta_p a}+\frac{\beta_p l}{\sinh^2\beta_p l}, \qquad \zeta_p = \coth\gamma_p h+\frac{\gamma_p h}{\sinh^2\gamma_p h} \tag{2.16}$$

and the relations

$$\frac{l+(1-\nu^2)a}{\nu l}B_1 = \sum_{k=1}^{\infty}(Y_k+Z_k)-\frac{1}{\nu h}\sum_{k=1}^{\infty}\left(\frac{a_k}{\alpha_k}-\frac{e_k}{\gamma_k}\right)-\frac{1}{l}\sum_{k=1}^{\infty}\frac{(-1)^k c_k}{\beta_k}+$$
$$+\frac{E}{2l}\left(\frac{a}{h}b_0+\frac{d_0}{\nu}+\frac{l}{E}f_0\right), \qquad B_2 = \frac{1}{h}\sum_{k=1}^{\infty}\left(\frac{a_k}{\alpha_k}-\frac{e_k}{\gamma_k}\right)+B_1 \tag{2.17}$$

From (2.2), according to (2.7), we obtain

$$b_1{}^* = b_2{}^* = -\frac{\nu a}{2h} b_0 + (1-\nu^2)\frac{a}{E} B_1 \qquad (2.18)$$

Substituting (2.17) and (2.18) into the third equality of (2.2), we find that

$$A_2 = \frac{1}{\nu h}\sum_{k=1}^{\infty}\left(\frac{a_k}{\alpha_k} - \frac{e_k}{\gamma_k}\right) - \frac{E}{2l}\left(\frac{a}{h} b_0 + \frac{d_0}{\nu}\right) + \frac{l + (1-\nu^2) a}{\nu l} B_1 \qquad (2.19)$$

Thus the coefficients $A_k^{(1)}$ and $B_k^{(1)}$ can be determined directly from (2.3). The constants $b_1{}^*$, $b_2{}^*$, A_1, A_2 and B_2 can be expressed according to (2.2) and (2.17) to (2.19) in terms of B_1. The coefficients $C_k{}^{(1)}, D_k{}^{(1)}, E_k{}^{(1)}, F_k{}^{(1)}, A_k{}^{(2)}, C_k{}^{(2)}$ and $F_k{}^{(2)}$ can be expressed on the basis of (2.4) to (2.7) and (2.8) in terms of X_k, Y_k and Z_k, for the determination of which we have the infinite systems of linear equations (2.9) and (2.10) with absolute terms dependent on B_1. The coefficient B_1 can be determined from (2.17) using the solutions of (2.9) and (2.10).

3. We shall now prove that the set of infinite systems (2.9) and (2.10) is quasi-completely regular. The sum of the absolute values of the coefficients of equations (2.9) is given by

$$\sum_{k=1}^{\infty} |a_{pk}| + \sum_{k=1}^{\infty} |b_{pk}| = \frac{2\gamma_p}{h\zeta_p}\left[\sum_{k=1}^{\infty}\frac{1}{\beta_k{}^2+\gamma_p{}^2} + \sum_{k=1}^{\infty}\frac{|\beta_k{}^2-\gamma_p{}^2|}{(\beta_k{}^2+\gamma_p{}^2)^2}\right] =$$
$$= \frac{2x_p}{\pi\zeta_p(x_p)}\left[\sum_{k=1}^{\infty}\frac{1}{k^2+x_p{}^2} + \sum_{k=1}^{\infty}\frac{|k^2-x_p{}^2|}{(k^2+x_p{}^2)^2}\right] \qquad (p = 1, 2, \ldots) \qquad (3.1)$$

where

$$x_p = \frac{\gamma_p h}{\pi} = p\frac{h}{l} \qquad \zeta_p(x_p) = \coth x_p\pi + \frac{x_p\pi}{\sinh^2 x_p\pi} \qquad (3.2)$$

$$\sum_{k=1}^{\infty} |a_{pk}| + \sum_{k=1}^{\infty} |b_{pk}| =$$
$$= \frac{2x_p}{\pi\zeta_p(x_p)}\left[\sum_{k=1}^{\infty}\frac{1}{k^2+x_p{}^2} \mp \sum_{k=1}^{k_p{}^\circ}\frac{k^2-x_p{}^2}{(k^2+x_p{}^2)^2} + \sum_{k_p{}^\circ+1}^{\infty}\frac{k^2-x_p{}^2}{(k^2+x_p{}^2)^2}\right] \qquad (3.3)$$

Here $k_p{}^\circ$ is an integer defined for the integers $k \leqslant k_p{}^\circ$ by the inequality

$$k^2 - x_p{}^2 \leqslant 0 \qquad (3.4)$$

From (3.4) we obtain

$$k_p{}^\circ \leqslant x_p \qquad (3.5)$$

From (3.5) we have, for the expression (3.3),

$$\sum_{k=1}^{\infty} |a_{pk}| + \sum_{k=1}^{\infty} |b_{pk}| = \frac{2x_p}{\pi\zeta_p(x_p)}\left[\sum_{k=1}^{\infty}\frac{1}{k^2+x_p{}^2} + \sum_{k=1}^{\infty}\frac{k^2-x_p{}^2}{(k^2+x_p{}^2)^2} - 2\sum_{k=1}^{k_p{}^\circ}\frac{k^2-x_p{}^2}{(k^2+x_p{}^2)^2}\right] =$$
$$= \frac{2x_p}{\pi\zeta_p(x_p)}\left[\frac{\pi}{2x_p}\left(\coth x_p\pi - \frac{1}{x_p\pi}\right) + \frac{\pi}{2x_p}\left(1 - \frac{x_p{}^2\pi^2}{\sinh^2 x_p\pi}\right)\frac{1}{x_p\pi} + 2x_p{}^2\sum_{k=1}^{k_p{}^\circ}\frac{1}{(k^2+x_p{}^2)^2} -\right.$$
$$\left. - 2\sum_{k=1}^{k_p{}^\circ}\frac{k^2}{(k^2+x_p{}^2)^2}\right] \leqslant \frac{1}{\zeta_p(x_p)}\left[\coth x_p\pi - \frac{x_p\pi}{\sinh^2 x_p\pi} + \frac{1}{x_p\pi}\,\frac{1-x_p{}^2}{1+x_p{}^2} + \right. \qquad (3.6)$$

$$+\frac{2}{\pi}\left(1+\tan^{-1}\frac{1}{x_p}\right)\Bigg]=f(x_p)$$

Here we have used the following sums and inequalities [15]:

$$\sum_{k=1}^{\infty}\frac{1}{k^2+x_p^2}=\frac{\pi}{2x_p}\left(\coth x_p\pi-\frac{1}{x_p\pi}\right) \tag{3.7}$$

$$\sum_{k=1}^{\infty}\frac{1}{(k^2+x_p^2)^2}=\frac{\pi}{4x_p^3}\left(\coth x_p\pi+\frac{x_p\pi}{\sinh^2 x_p\pi}-\frac{2}{x_p\pi}\right) \tag{3.8}$$

$$\sum_{k=1}^{\infty}\frac{k^2}{(k^2+x_p^2)^2}=\frac{\pi}{4x_p}\left(\coth x_p\pi-\frac{x_p\pi}{\sinh^2 x_p\pi}\right) \tag{3.9}$$

$$\sum_{k=1}^{k_p^\circ}\frac{1}{(k^2+x_p^2)^2}\leqslant\int_0^{k_p^\circ}\frac{dk}{(k^2+x_p^2)^2}\leqslant\frac{1}{4x_p^3}\left(1+\frac{\pi}{2}\right) \tag{3.10}$$

$$\sum_{k=1}^{k_p^\circ}\frac{k^2}{(k^2+x_p^2)^2}\geqslant\int_1^{k_p^\circ}\frac{k^2dk}{(k^2+x_p^2)^2}\geqslant\int_1^{x_p}\frac{k^2dk}{(k^2+x_p^2)^2}-\frac{1}{4x_p^2}=$$

$$=\frac{1}{4x_p}\left(\frac{\pi}{2}-1-\frac{1}{x_p}\frac{1-x_p^2}{1+x_p^2}-2\tan^{-1}\frac{1}{x_p}\right) \tag{3.11}$$

We find from (3.2) and (3.6) that

$$\lim_{x_p\to\infty}f(x_p)=1+\frac{2}{\pi} \tag{3.12}$$

For the second system of (2.10) we obtain

$$\sum_{k=1}^{\infty}|c_{pk}|=\frac{4\beta_p}{l\xi_p}\left[\sum_{k=1}^{\infty}\frac{\gamma_k^2}{(\gamma_k^2+\beta_p^2)^2}+\frac{\eta_p}{2\xi_p}\sum_{k=1}^{\infty}\frac{1}{\gamma_k^2+\beta_p^2}\right]=\frac{4y_p}{\pi\xi_p(y_p)}\left[\sum_{k=1}^{\infty}\frac{k^2}{(k^2+y_p^2)^2}+\right.$$

$$\left.+\frac{1}{2}\frac{\eta_p(y_p)}{\xi_p(y_p)}\sum_{k=1}^{\infty}\frac{1}{k^2+y_p^2}\right]=\frac{1}{\xi_p(y_p)}\left[\coth y_p\pi-\frac{y_p\pi}{\sinh^2 y_p\pi}+\frac{\eta_p(y_p)}{\xi_p(y_p)}\left(\coth y_p\pi-\frac{1}{y_p\pi}\right)\right]=$$

$$=\varphi(y_p) \tag{3.13}$$

where (3.14)

$$y_p=\frac{\beta_p l}{\pi}=p\frac{l}{h},\qquad \xi_p(y_p)=\coth y_p\pi+\coth\frac{y_p\pi a}{l},\qquad \eta_p(y_p)=\frac{y_p\pi}{\sinh^2 y_p\pi}+\frac{y_p\pi a/l}{\sinh^2(y_p\pi a/l)}$$

From (3.13) and (3.14) we find that

$$\lim\varphi(y_p)=1/2\qquad (y_p\to\infty) \tag{3.15}$$

For the sum of the absolute values of the coefficients of the third equation of (2.10) we have

$$\sum_{k=1}^{\infty}|d_{pk}|=\frac{2\beta_p}{l\xi_p}\sum_{k=1}^{\infty}\frac{1}{\gamma_k^2+\beta_p^2}=\frac{2y_p}{\pi\xi_p(y_p)}\sum_{k=1}^{\infty}\frac{1}{k^2+y_p^2}=$$

$$=\frac{1}{\xi_p(y_p)}\left(\coth y_p\pi-\frac{1}{y_p\pi}\right)=\psi(y_p) \tag{3.16}$$

From the inequalities

$$\coth y_p\pi - \frac{1}{y_p\pi} \leqslant 1, \qquad \coth y_p\pi + \coth\frac{y_p\pi a}{l} \geqslant 2 \quad \text{for} \quad 0 \leqslant y_p\pi \leqslant \infty \tag{3.17}$$

and from Formula (3.16) we obtain

$$\psi(y_p) \leqslant {}^1/_2 \qquad \text{for} \;\; 0 \leqslant y_p \leqslant \infty \tag{3.18}$$

Thus, from (3.12), (3.15) and (3.18) we have the following estimates for the sum of the moduli of the coefficients of each line of the infinite system (2.10) for $0 \leqslant y_p \leqslant \infty$

$$\lim_{x_p\to\infty}\left\{\sum_{k=1}^{\infty}|a_{pk}| + \sum_{k=1}^{\infty}|b_{pk}|\right\} \leqslant 1 + \frac{2}{\pi}, \qquad \lim_{y_p\to\infty}\sum_{k=1}^{\infty}|c_{pk}| = \frac{1}{2}, \qquad \sum_{k=1}^{\infty}|d_{pk}| \leqslant \frac{1}{2} \tag{3.19}$$

Substituting Y_p and Z_p from the second and third systems of (2.10) into the first, and X_p from the first into the remaining ones, we obtain an infinite system in the unknowns X_p, Y_p and Z_p

$$\begin{aligned} X_p &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(a_{pk}c_{kn} + b_{pk}d_{kn})\,X_n + \sum_{k=1}^{\infty}(a_{pk}n_k + b_{pk}r_k) + m_p \\ Y_p &= \sum_{k=1}^{\infty}c_{pk}\left(\sum_{n=1}^{\infty}a_{kn}Y_n + \sum_{n=1}^{\infty}b_{nk}Z_n\right) + \sum_{k=1}^{\infty}c_{pk}m_k + n_p \\ Z_p &= \sum_{k=1}^{\infty}d_{pk}\left(\sum_{n=1}^{\infty}a_{kn}Y_n + \sum_{n=1}^{\infty}b_{kn}Z_n\right) + \sum_{k=1}^{\infty}d_{pk}m_k + r_p \end{aligned} \qquad (p = 1, 2, \ldots) \tag{3.20}$$

For the sums of the absolute values of the coefficients of system (3.20), according to (3.19), we have the following estimates:

$$\begin{aligned} &\lim_{\substack{x_p\to\infty\\ y_p\to\infty}}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|a_{pk}c_{kn} + b_{pk}d_{kn}| \leqslant \lim_{\substack{x_p\to\infty\\ y_p\to\infty}}\sum_{k=1}^{\infty}|a_{pk}|\sum_{n=1}^{\infty}|c_{kn}| + \lim_{\substack{x_p\to\infty\\ y_p\to\infty}}\sum_{k=1}^{\infty}|b_{pk}|\sum_{n=1}^{\infty}|d_{kn}| = \\ &= \lim_{\substack{x_p\to\infty\\ y_p\to\infty}}\sum_{n=1}^{\infty}|c_{kn}|\left(\sum_{k=1}^{\infty}|a_{pk}| + \sum_{k=1}^{\infty}|b_{pk}|\right) = \lim_{\substack{x_p\to\infty\\ y_p\to\infty}}\sum_{n=1}^{\infty}|d_{kn}|\left(\sum_{k=1}^{\infty}|a_{pk}| + \sum_{k=1}^{\infty}|b_{pk}|\right) \leqslant \\ &\leqslant \frac{1}{2}\left(1 + \frac{2}{\pi}\right) = 0.5\cdot 1{,}6366 = 1 - \theta \end{aligned} \tag{3.21}$$

$$\lim_{\substack{x_p\to\infty\\ y_p\to\infty}}\sum_{k=1}^{\infty}|c_{pk}|\left(\sum_{n=1}^{\infty}|a_{kn}| + \sum_{n=1}^{\infty}|b_{kn}|\right) \leqslant \frac{1}{2}\left(1 + \frac{2}{\pi}\right) = 0.5\cdot 1.6366 = 1 - \theta$$

$$\lim_{\substack{x_p\to\infty\\ y_p\to\infty}}\sum_{k=1}^{\infty}|d_{pk}|\left(\sum_{n=1}^{\infty}|a_{kn}| + \sum_{n=1}^{\infty}|b_{kn}|\right) \leqslant \frac{1}{2}\left(1 + \frac{2}{\pi}\right) = 0.5\cdot 1.6366 = 1 - \theta$$

$$(\theta = 0.1817)$$

Note that the estimates (3.19) which were derived for $x_p \to \infty$ and $y_p \to \infty$ are already valid, for $x_p \geqslant 5$ and $y_p \geqslant 5$.

In addition, from (3.6), (3.13) and (3.16) we easily see that

$$\lim f(x_p) = 1/2 \quad \text{as} \quad x_p \to 0, \qquad \lim \varphi(y_p) = \lim \psi(y_p) = 0 \quad \text{as} \quad y_p \to 0$$

and the sums of absolute values of the coefficients of systems (3.20) tend to zero. This enables us to obtain an approximate solution for the infinite systems with a high degree of accuracy.

Thus, on the basis of (3.21) the infinite system (2.10) is, for arbitrary ratio of a, l and h and any possible value of Poisson's ratio, quasi-completely regular [11]. As can be seen from (2.11) to (2.15) and (3.20) the absolute terms of the infinite systems (2.10) and (3.20) are of the order of the Fourier coefficients of expansions (1.9) to (1.11). Consequently, they are finite and of the order of not less than p^{-1} if the external load and the first derivatives of the displacements $v^{-1}(x, h)$ and $u^{2}(l, y)$ are, in the given range, piecewise continuous.

The quasi-complete regularity of the infinite system (2.10), together with the finiteness of the absolute terms in (2.13) to (2.15), enables us to evaluate the required coefficients of the expansions of $\Phi(x, y)$ (1.4) and (1.5) with any degree of accuracy [14]. On this basis we can evaluate the upper and lower bounds for the stresses σ_x, σ_y and τ_{xy} and the displacements u and v.

Note that the series which define the required stresses and displacements are divergent on the sides of the rectangle. In order, therefore, to determine the nature of the variation of these quantities (and this includes the pressure under the die) it is necessary to evaluate them at internal points very close to the boundary [14] where these series converge as geometrical progressions.

The solution of the contact problem for a rectangle when only the stresses are specified everywhere on the boundary outside the contact area, together with numerical examples illustrating the effect of external load and ratios of the dimensions of the rectangle to the length of contact, will be the subject of a later paper by the authors.

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